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Voting Schemes for which It Can Be Difficult to Tell Who Won the Election*

J. Bartholdi, III, C. A. Tovey, and M. A. Trick**

School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA

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Abstract. We show that a voting scheme suggested by Lewis Carroll can be impractical in that it can be computationally prohibitive (specifically, *NP*-hard) to determine whether any particular candidate has won an election. We also suggest a class of "impracticality theorems" which say that any fair voting scheme must, in the worst-case, require excessive computation to determine a winner.

1. Introduction

We can consider a voting scheme to be a well-defined rule by which, given any input consisting of a set C of candidates and a set V of transitive preference orders on C (the preferences of the electorate), one can determine a subset of C whose elements are the winners (allowing for ties). When a voting scheme is considered to be a rule to determine the winner(s) of an election, it is natural to ask about the computational resources required by the scheme. For example, can the scheme be guaranteed to identify a winner quickly? In other words, is there an efficient algorithm to find a winner under the given voting scheme?

For both practical and theoretical reasons, an algorithm is considered formally efficient if it requires a number of computational steps that is at most polynomial in the size of the problem. Problems for which there are polynomial-time algorithms are generally considered to be tractable, and those which can require exponential time to solve are considered inherently intractable.

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Computational complexity can classify voting schemes based on a well-studied hierarchy of complexity classes that are thought to be distinct. For example, within this hierarchy is the problem class *NP*, which consists of those questions for which a "yes" answer can be justified in polynomial time. The hardest problems in *NP* are known as "*NP*-complete", and all such problems are equivalent in the sense that any problem in *NP* can be reworded as an instance of an *NP*-complete problem within polynomial time. Thus the existence for a polynomial-time algorithm to solve a single *NP*-complete problem implies that all problems in *NP* can be solved in polynomial time. That no one has found such an algorithm is taken as strong circumstantial evidence that *NP*-complete problems are inherently intractable. (For more on complexity and *NP*-completeness, see [7].)

We will show that, for two voting schemes, several natural questions about the outcome of an election are *NP*-complete. This suggests that it is very unlikely that one can find an efficient algorithm to answer them. Thus these questions can be too difficult to answer (at least for "sufficiently large" elections without special structure), and so the voting schemes might be impractical.

2. The Difficulty of Tabulating Scores

We assume throughout that the preferences of all voters are strict (irreflexive and antisymmetric), transitive, and complete. We formalize as follows the essential question to be answered by any voting scheme.

Election Winner

Instance: Set of candidates C, and one distinguished member c of C; set V of preference orders on C.

Question: Is c a winner under the specified voting scheme?

In most historical voting schemes only polynomial time is required to answer this question (a practical necessity when counting paper ballots!). For example, to solve *Plurality Winner* requires only $O(|V|+|C|)$ work to count first-place votes and identify the candidate with the most. It is unusual then to discover voting schemes that can apparently require exponential time to tell whether any particular candidate has won the election. We exhibit two such schemes. One was invented by the mathematician Charles Dodgson (better known as Lewis Carroll), and the other was suggested by J. Kemeny. We show that under a either a Dodgson election or a Kemeny election, it is *NP*-hard (that is, at least as hard as an *NP*-complete problem) to determine whether any particular candidate has won! Thus these schemes are capable of taking an impractically long time to determine a winner. Others have observed the empirical difficulty of computing winners under these schemes [for example, 6]. However, we establish this in a formal sense, and suggest computational complexity as another aspect to be taken into consideration when practical voting schemes are to be judged.

The rationality criterion that troubled Dodgson was the famous one first formalized by the Marquis de Condorcet [4], which requires that a voting scheme elect any candidate (the "Condorcet winner") that would defeat any other candidate in a pairwise election with simple vote counts. Condorcet observed that there can be elections in which no candidate is a Condorcet winner (the "Phenomenon of Cyclic Majorities"). Accordingly, Dodgson sought a voting scheme that would still choose "rationally" in the absence of a Condorcet winner. He suggested the following voting scheme (reprinted at length in [3]; summarized in [13]).

The Dodgson Voting Scheme

A Dodgson winner is a candidate who is "closest" to being a unique Condorcet winner, where by "closest" we mean the following: Imagine that an election supervisor is empowered to change the ballot of any voter through pairwise interchange of candidates adjacent in the voter's preference order. Then a Dodgson winner is a candidate who requires the fewest interchanges to become a Condorcet winner. (Hereafter we will refer to such pairwise interchanges simply as "switches".) The minimum number of switches for a candidate to become a Condorcet winner is the Dodgson score of that candidate. A candidate with the smallest Dodgson score is a winner of the election.)

As an example of the Dodgson scheme, consider the candidates A, B, C, D, and three voters, one with each of the preferences $A>D>B>C$, $B>C>A>D$, and $C>A>D>B$. Then the Dodgson score of A is 1, since switching C and A in the preference $C>A>D>B$ is sufficient to make A the Condorcet winner, and no fewer switches will do so. Similarly, the Dodgson score of B is 2, since at least 2 switches are necessary to beat both A and D, and this can be accomplished by switching B to the top of the preference $A>D>B>C$. The Dodgson score of C is 1, since C can be made to defeat B by a single switch within the preference $B>C>A>D$ to become a Condorcet winner. Finally, the Dodgson score of D is 4 since at least 3 switches are required to defeat A (one in each preference), and a additional switch is required for D to defeat C. Thus A and C tie for Dodgson winner of this election.

Dodgson described a winner of the election, but did not specify an algorithm to identify a winner. We show that any conceivable algorithm to do this apparently must require excessive time, at least in the worst-case.

In showing that it can be difficult to tell any candidate whether he has won the election, we develop two supporting lemmas. The first and key lemma says that it can be difficult to tell whether a candidate did well. We formalize this question as

Dodgson Score

Instance: Set of candidates C, and a distinguished member c of C; set V of preference orders on C; a positive integer K.

Question: Is the Dodgson score of candidate c less than or equal to K?

Lemma 1. *Dodgson score is NP-complete.*

Proof. First observe that *Dodgson score* is in *NP*, since a "yes" answer can be justified in polynomial time by identifying appropriate switches and tabulating the vote.

Now we contrive an election for which determining the winner entails solving *Exact Cover by 3-Sets (X3C)*, which is known to be *NP*-complete [7].

Exact Cover by 3-Sets (X3C)

Instance: Set B with $|B|=3q$ and a collection S of 3-element subsets of B.

Question: Does S contain an exact cover for B, i.e., a subcollection S' of S such that every element of B occurs in exactly one member of S'?

First we define the set C of candidates: For each element b_i of B, create two corresponding candidates b_i, and f_i. Similarly, we create a candidate s_j for each S_j in S.

Now we devise the set V of voters (which are identified with their preferences). V consists of the following subsets:

1. Swing Voters. Create voters corresponding to members of S: For each subset $S_j=\{b_{j1}, b_{j2}, b_{j3}\}$ create a single voter $(b_{j1}>b_{j2}>b_{j3}>s_j>c>\ldots)$, where the entries after c are in arbitrary order. We call these voters "swing voters", since their votes will be crucial to the result of the election. Note that switching c up 1 position in such a preference order gains 0 votes against members of B; switching 2 times gains 1 vote; switching 3 times gains 2 votes; switching 4 times, so that c is at the very top of the preference order, gains 3 votes against members of B in pairwise elections. Thus, among swing voters, to get additional votes for c over members of B requires at least 4/3 switches per vote on the average, and to achieve this, any voter who switches at all must switch c upward 4 times, to the very top of his preference order.

The swing voters are the means by which we embed $X3C$ in *Dodgson Score*. However we must pad the elctorate with additional voters to make sure that this embedding captures all the difficulty of $X3C$:

2. Equalizing Voters. Let N_i be the number of votes from swing voters that b_i would get in a pairwise election against c; let $N_{\max}$ be the largest N_i. For each b_i create $N_{\max}-N_i$ additional (identical) voters $(b_i>f_i>c>\ldots)$ so that each b_i would get exactly $N_{\max}$ votes in a pairwise election against c. We call these "equalizing voters" since they make all the b_i score equally well against c among swing voters and equalizing voters. Among equalizing voters, to get votes for c over members of B requires at least 2 switches per vote on the average.

3. Incremental Voters. Finally create a class of identical voters $(b_1>\ldots>b_{|B|}>f_1>\ldots>f_{|B|}>c>\ldots)$ sufficient in number so that any candidate b_i would defeat c by exactly 1 vote. Among incremental voters, to get votes for c over members of B requires at least 2 switches per vote on the average.

Now that we have defined the election, consider whether c can be made a Condorcet winner by no more than $4|B|/3$ switches. If he can, c must convince the electorate to prefer him to each of the b_i; but this requires at least $4|B|/3$ switches, and is achievable only if *i)* all switches are among swing voters, and *ii)* each swing voter makes 4 switches, to move c to the top of his preferences. But any set of swing voters that can elect c by making no more than $4|B|/3$ switches corresponds to an exact 3-cover of B by members of S. □

The second supporting lemma says that it can be hard to compare the scores of two candidates, which problem we formalize as

Election Ranking

Instance: Set of candidates C, and distinguished members c, c' of C; set V of preference orders on C.

Question: Did c defeat c' in the election?

Lemma 2. *Dodgson ranking is NP-hard.*

Proof. We use the construction from the proof of Lemma 1 with the additional properties that the total number of voters is odd and there exists at least 1 equalizing voter.

To get the total number of voters odd, it suffices to get $|B|$ odd and $|S|$ even. If $|B|$ is not odd, add 3 new artificial base elements, a_1, a_2, and a_3, and add the set (a_1, s_2, a_3) to S. If S is not now even, choose any member of S and add another copy of it. If there are no equalizing voters in the election corresponding to this enlarged instance of $X3C$, then choose any element of S and add 2 copies of it; this preserves the parity of S, and ensures the existence of at least 1 equalizing voter. These changes to the instance of $X3C$ are merely cosmetic, since whether there exists a 3-cover remains invariant under these modifications.

Now enlarge the election by adding a new candidate c' to everyone's preference orders. To do this, first arbitrarily choose some equalizing voter v to be special. Divide the remaining voters into two arbitrary groups of equal size. All the voters in one group insert c' at the very top of their preferences; all the voters in the other group insert c' at the very bottom of their preferences. The special voter v inserts c' in position $1+4|B|/3$ in his preference order.

The Dodgson score of c' is no more than $4|B|/3$: without the vote of v, c' must tie every other candidate in pairwise elections, so that making $4|B|/3$ switches to move c' to the top of v's preferences will enable c' to defeat all the other candidates. On the other hand, the Dodgson score of c' must be at least $4|B|/3$ since at least that many candidates defeat him in pairwise elections (exactly those candidates preferred to him by v). Thus the Dodgson score of c' is exactly $4|B|/3$. The proof now follows analogously to the proof of Lemma 1. Thus *Dodgson ranking* is as hard as an *NP*-complete problem; but since we do not know whether *Dodgson ranking* is in *NP*, we can say only that it is *NP*-hard. □

Theorem 1. *Dodgson winner is NP-hard.*

Proof. It is straightforward but tedious to pad the contrived election (from Lemmas 1 and 2) with a polynomial number of additional voters so that c and c' (and no others) are tied for first place. □

We think Lewis Carroll would have appreciated the idea that a candidate's mandate might have expired before it was ever recognized.

3. Efficiency at the Cost of Universality

The Dodgson scheme can be made at least formally efficient if we place an a priori restriction on the size of the elections in which it will be employed. For example, if we bound in advance the number of voters, then we can determine the Dodgson winner in polynomial time by enumeration: Any particular candidate c can be permuted to at most $|C|$ different positions in the preference order of a voter, so there are at most $|C|^{|V|}$ possible ways of placing c in the preferences of the electorate. We can count the number of switches implicit in each of these ways that lead c to be a Condorcet winner, and the fewest number of switches is the Dodgson score of c. Finally we can compare scores and choose the smallest. If $|V|$ is bounded by a constant, then this procedure is technically polynomial-time (even though this might not be reassuring for large $|V|$).

Similarly, if we bound in advance the number of candidates, the Dodgson scheme can be made to run in polynomial time by solving an integer linear program with a very large but fixed number of variables and constraints. The problem of determining the Dodgson score of candidate c can be formulated as an integer linear program in the following way. Index by i the types of preference orders found among the voters, and let N_i be the number of voters of type i. Let x_{ij} be the number of voters with preferences of type i for which candidate c will be moved upwards by j positions. Let e_{ijk} be 1 if the result of moving candidate c by j positions upward in a preference order of type i is that c gains an additional vote against candidate k, and 0 otherwise. Let d_k be the deficit of c with respect to candidate k, that is, the minimum number of votes that c must gain against k to defeat him in a pairwise election. If c already defeats k, then $d_k=0$. Then the Dodgson score of c is the value of integer linear program.

$$\min \Sigma_{ij} j x_{ij} \text{ subject to} \tag{3.1}$$

$$\Sigma_j x_{ij} = N_i \qquad \text{(all types } i \text{ of preference orders)}$$

$$\Sigma_{ij} e_{ijk} x_{ij} \geqq d_k \qquad \text{(all candidates } k\text{)}$$

$$x_{ij} \geqq 0 \ , \qquad \text{integer.}$$

The first set of constraints restricts the numbers and types of preferences to those actually present among the voters, and the second set of constraints ensures that c will become a Condorcet winner. The objective is to minimize the number of switches.

The number of different types of voters (preference orders) is no greater than $|C|!$, and the number of different positions in any preference order is $|C|$. Consequently there are no more than $|C| \times (|C|!)$ variables x_{ij} and no more than $|C|!+|C|$ non-trivial constraints. If we limit the applicability of the Dodgson scheme by restricting $|C|$ to be no larger than some prespecified number, (3.1) is polynomially solvable, at least by the algorithm of Lenstra [12], for which the time bound, though potentially enormous, is technically a polynomial.

The effort required to determine a Dodgson winner appears to increase more quickly as a function of $|C|$ than as a function of $|V|$. Even in real elections $|C|$ can be large enough to make the Dodgson scheme potentially impractical: the *New York Times* of 1 April 1986 reported 20 candidates for mayor of Tulsa, Oklahoma!

4. An "Impracticality Theorem"

Kemeny [10, 11] has suggested another voting scheme that extends the Condorcet principle. He defined the outcome of an election to be a consensus ranking of the alternatives, and suggested that the consensus be a preference order that minimizes the sum of "distances" to the preferences of the voters. We show that it can be difficult to determine the outcome of an election under Kemeny scoring. As a corollary we conclude that every voting rule that satisfies certain modest fairness criteria must be inefficient at determining a winner.

(We thank the editor and referees for pointing out that the difficulty of scoring a Kemeny election has been independently established by others, including J. Orlin (private correspondance), and, most notably, Wakabayashi [16], who comprehensively analyzed the complexity of median and mean procedures. In addition, the complexity of related problems has been discussed elsewhere (for example [1], [8]).)

Kemeny defined the "distance" between two preferences P and P' as $\mathrm{dist}(P, P') = \Sigma d(j, k)$, where the sum is taken over all unordered pairs of candidates j and k, and where $d(j, k) = 0$ if P and P' agree on candidates j and k; $d(j, k) = 2$ if P prefers j to k but P' prefers k to j; and $d(j, k) = 1$ if P prefers j to k but P' is indifferent between j and k. A Kemeny consensus is a preference that minimizes $\Sigma N_i \,\mathrm{dist}(P, P_i)$, where N_i is the number of voters with preference P_i.

We will need the following technical result, which enables us to consider only strict preferences.

Lemma 3. *If all voter preferences are strict, then there exists some Kemeny consensus that is strict.*

Proof. Let P be a Kemeny consensus that includes ties (indifference), and consider any set T of candidates that are mutually tied under P. Let c be any candidate in T and compute $A = \Sigma |\{\text{voters who prefer } c \text{ to } d\}|$ and $B = \Sigma |\{\text{voters who prefer } d \text{ to } c\}|$, where the sums are taken over all d in T. If $A > B$, then the Kemeny score of P could be improved by breaking ties in favor of c; similarly, if $A < B$, the Kemeny score could be improved by preferring the remaining candidates of T to c. But since P is a Kemeny consensus, its score must be minimum, so that $A = B$. Finally, since $A = B$, we can break ties arbitrarily in favor of c to produce a new preference order with fewer ties, but with the same minimum Kemeny score. Repeated application of this produces a Kemeny consensus with no ties. □

Theorem 2. *Kemeny score is NP-complete, and Kemeny ranking and Kemeny winner are NP-hard.*

Proof. Kemeny score is in *NP* since the score of any candidate can be computed in polynomial time.

We show the problem is hard by showing that the following problem, which is known to be *NP*-complete [7], can be polynomially transformed to it.

Feedback Arc Set

Instance: Directed graph G, with vertices C; positive integer K.

Question: Is there a subset of no more than K arcs which includes at least one arc from every cycle in G?

For any instance of *Feedback Arc Set*, interpret G as representing the outcomes of pairwise contests between candidates C. By [15], G is realizable by a set V of voters for wich $|V|$ is even and "small" (polynomial in $|C|$), whose preferences are all strict, and whose preferences decide each contest by exactly 2 votes. Thus, for any arc (i,j) of G, exactly $(|V|+2)/2$ voters prefer candidate i to j, and $(|V|-2)/2$ voters prefer candidate j to i. Therefore any strict preference P must disagree with at least $(|V|-2)/2$ voters on the relative ranking of candidates i and j, and so must incur a "fixed-cost" of $(|V|-2)$ to its Kemeny score. If P disagrees with the majority of voters and prefers j to i, then there is an additional penalty of (2 voters)(2 points/voter)$=4$ points. Since this holds for each of the $|V|(|V|-1)/2$ pairs of candidates, the Kemeny score of P must be at least $|V|(|V|-1)(|V|-2)/2$, plus 4 times the number of contests in which P disagreed with the majority.

Now consider the question whether there exists a consensus whose Kemeny score is no larger than $|V|(|V-1|)(|V|-2)/2)+4K$. By Lemma 3, the answer is "yes" if-and-only-if there exists a strict preference with the same Kemeny score. By construction, a strict consensus with this score must agree with the majority on all but K of the pairwise contests, and the arcs corresponding to these pairwise contests are a feedback arc set for G. Therefore *Kemeny score* is *NP*-complete.

As for the Dodgson scheme, the election can be padded to establish that *Kemeny ranking* and *Kemeny winner* are both *NP*-hard. □

Now we reword Theorem 2 in a provocative way. Following [17] we define a voting scheme to be neutral if it is symmetric in its treatment of candidates; to be Condorcet if it elects any Condorcet winner; to be consistent if, when two disjoint subsets of the electorate, voting separately, arrive at the same consensus, then their voting together always produces this same consensus. Young and Levenglick [17] proved that Kemeny scoring is the unique voting scheme that is neutral, consistent, and Condorcet. Hence we have the following.

Corollary. *Under any voting scheme that is neutral, consistent, and Condorcet, the Winner Problem is NP-hard.*

Since only the Kemeny rule satisfies the hypotheses, this corollary is not entirely satisfying. Nevertheless, it can be taken as a model for a new type of impossibility theorem (that might be called an "impracticality theorem"), the general form of which is "Fair elections are impractical". Are there stronger versions of this theorem?

5. Conclusions

Many theorems and some practical experience attest that any conceivable voting scheme is capable of some form of unacceptable behavior, such as violating formalized notions of fairness or rationality (for example [5, 9, 13]; summary in [14]). We remark that the computational complexity of the corresponding election winner problem is also an important aspect one should consider when judging voting schemes.

It seems desirable for a voting scheme to be dependably quick in its decisions. It would be interesting to explore the extent to which this is reconcilable with notions of fairness. In particular, do there exist more potent impracticality theorems than the one we offer?

In addition, there are other computational aspects of voting to be explored. For example, elsewhere we have exhibited a voting scheme that is easy to operate, but is computationally resistant to manipulation [2].

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